Conditionally averaged vorticity field and turbulence modeling

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The conditionally averaged vorticity (CAV) field with fixed vorticity in a point is obtained from a direct numerical simulation of isotropic turbulence. The characteristic attenuation scale for the twisting and hyperboloidal CAV components is found to be of order ten times greater than the Kolmogorov microscale. A simple analytical model qualitatively agrees with the obtained CAV. For turbulent free-surface flows, the twisting part of CAV is expected to connect to the free surface. An alternative type of subgrid-scale modeling of turbulence, based on CAV, is suggested for the large-eddy simulations.

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I. INTRODUCTION

The major goal of the theory of turbulence as driven by various engineering disciplines (including energy, naval, aerospace, and environmental) is to reduce the enormous number of degrees of freedom in turbulent flows to a level that is manageable by computer simulations. In order to do this properly, we have to understand what kind of statistical structures are naturally created in small-scale turbulent motion and by a careful subgrid-scale analysis adjust numerics to model these structures.

The most important effect (physically and numerically) in three-dimensional motion is vortex stretching, which is statistically balanced with viscous smoothing [1–5]. The corresponding statistical structure is represented by the conditionally averaged vorticity field (CAV) [3] as function of distance $\mathbf{r}$ from a point with fixed vorticity $\boldsymbol{\omega}$:

$$\overline{\Omega}_i(\mathbf{r},\boldsymbol{\omega})=[f_1(\omega)]^{-1}\int \omega_i' f_2(\mathbf{r},\boldsymbol{\omega},\boldsymbol{\omega}')d^3\omega' \ , \tag{1}$$

where $\mathbf{r}=\mathbf{x}'-\mathbf{x}$ and f_1 and f_2 are one-point and two-point probability density functions (PDF's) of the vorticity field. For locally isotropic turbulence, f_1 depends only on absolute value of vorticity ω. For statistically nonstationary and nonhomogeneous turbulence all characteristics may parametrically depend on time and absolute position $\mathbf{x}$.

Conditional averaging of the Navier-Stokes equations (NSE), written in terms of vorticity field with fixed vorticity in a point, transforms the major nonlinear stretching term into a linear term [2,3]. This allows, in particular, an analytical study of CAV. We note that the conditionally averaged NSE with fixed vorticity in n points corresponds to a hierarchy of "kinetic" equations for the n-point PDF [2,3].

Based on the local isotropy of turbulence and the solenoidality of the vorticity field, we have a general expression for the Fourier transform of (1) [3]:

$$\widetilde{\Omega}_i(\mathbf{k},\boldsymbol{\omega})=g(k,\omega,\mu)(\sigma_i-\mu n_i)+h(k,\omega,\mu)\varepsilon_{ijk}\sigma_j n_k \ , \tag{2}$$

where $\sigma_i=\omega_i\omega^{-1}$, $n_i=k_ik^{-1}$, $\mu=\sigma_in_i$, $g(-\mu)=g(\mu)$, and $h(-\mu)=-h(\mu)$. $\mathbf{k}$ is the wave number vector with unit vector $\mathbf{n}$, $\boldsymbol{\sigma}$ is the unit vector of fixed vorticity, μ is the scalar product of these unit vectors, and ε_{ijk} is the unit antisymmetric tensor. Scalar g is a symmetric function of μ, and h is antisymmetric. Scalar h represents twisting of vortex lines, which is necessary for the statistical balance between vortex stretching and viscous smoothing for large Reynolds numbers [2–6]. ($\mathrm{Re}=Lv/\nu$, where L is the external scale, v is the characteristic velocity, and ν is the molecular viscosity.) This statistically important twist probably contributes to the helically shaped explosion of "vortex strings" that occurs when vortex tubes become unstable [4,7].

II. CONDITIONAL BALANCE OF VORTICITY

Conditional averaging of NSE leads to conditional balance of vorticity [3,5]:

$$\overline{\frac{\partial\omega}{\partial t}}=(\alpha-\beta)\omega, \quad \alpha(\omega,t)\equiv\overline{\frac{\partial v_i}{\partial x_k}}\sigma_i\sigma_k \ ,$$
$$\beta(\omega,t)\equiv-\nu\overline{\Delta\omega_i}\sigma_i \ . \tag{3}$$

Here the overbar denotes conditional averaging with fixed $\boldsymbol{\omega}$, α is the conditionally averaged rate of vortex stretching, β represents viscous smoothing, and v_i is the velocity field. We note that conditional averaging generally does not commute with spatial and temporal derivatives [3]. The convective term gives no contribution to the balance (3) for locally isotropic turbulence, and generally its contribution is small for large Reynolds numbers [3]. By expressing velocity in terms of an integral over vorticity for an incompressible fluid and after simple manipulations, we have [3]:

$$\alpha=-\varepsilon_{ijm}\sigma_i\int n_j\widetilde{\Omega}_i\mu d^3k=-\int\mu(1-\mu^2)hd^3k \ , \tag{4}$$

$$\beta\omega=\nu\sigma_i\int k^2\widetilde{\Omega}_id^3k=\nu\int k^2(1-\mu^2)gd^3k \ . \tag{5}$$

We see that vortex stretching is linked with the twisting of vortex lines, as represented by the scalar h. It was predicted [8,3] for large Reynolds numbers that the conditionally averaged vortex stretching term is balanced with

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viscous smoothing term: $\alpha(\omega,t)\approx\beta(\omega,t)$. This prediction was confirmed recently [5] by direct numerical simulation (DNS) that revealed an exponential behavior in the vortex stretching term:

$$\alpha\approx 0.13\omega_* \exp(0.16\omega/\omega_*), \quad \omega_*^2\equiv\langle\omega^2\rangle=\varepsilon/\nu . \tag{6}$$

Here ε is the mean rate of the energy dissipation and numerical coefficients in (6) are the same for the range $51.8\le R_\lambda\le 79.9$, where $R_\lambda\sim\sqrt{\mathrm{Re}}$ is the Reynolds number, based on the Taylor microscale.

The conditional balance (3) is more informative than the traditional unconditional balance:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\omega^2\rangle=\left\langle\frac{\partial v_i}{\partial x_k}\omega_i\omega_k\right\rangle-\nu\left\langle\left[\frac{\partial\omega_i}{\partial x_k}\right]^2\right\rangle . \tag{7}$$

Here ⟨ ⟩ means unconditional statistical averaging. If we multiply (3) by $\omega p(\omega,t)$, where $p=4\pi\omega^2 f_1$ is the PDF of the vorticity magnitude, and integrate over ω, we recover (7). Details of conditional and unconditional averaging are in Ref. [3]. It was found [5] that $p(\omega)\sim\exp(-1.8\omega/\omega_*)$ and $\exp(-2.1\omega/\omega_*)$ for $R_\lambda=79.9$ and 51.8, respectively for $\omega/\omega_*\ge 0.5$. The experimentally measured attenuation $p(\omega)\sim\exp(-2.56\omega/\omega_*)$ is more rapid than DNS predictions probably because of the limited range of ω that has been measured [9]. The negative exponent in $p(\omega)$ is of order of magnitude larger than the positive exponent in $\alpha(\omega)$, so the finite production of enstrophy is insured. We note that the energy spectra as predicted by DNS agrees very well with experimental measurements [5]. The coefficients $\alpha(\omega)$ and $\beta(\omega)$ provide partial information about the CAV field. The next step is to obtain the whole field, which requires more extensive work that is described in the next section.

III. CONDITIONALLY AVERAGED VORTICITY FIELD

The inverse Fourier transform of (2) gives

$$\bar{\Omega}_i(\mathbf{r},\boldsymbol{\omega})=a(r,\omega,\gamma)\sigma_i+b(r,\omega,\gamma)\lambda_i+c(r,\omega\gamma)\tau_i ,$$

where

$$\gamma=\rho_i\sigma_i, \quad \rho_i=r_ir^{-1}, \quad \lambda_i=(\rho_i-\gamma\sigma_i)\gamma_1^{-1} ,$$
$$\gamma_1=(1-\gamma^2)^{1/2}, \quad \tau_i=\varepsilon_{ijk}\sigma_j\lambda_k ,$$
$$a(-\gamma)=a(\gamma), \quad b(-\gamma)=-b(\gamma), \quad c(-\gamma)=-c(\gamma) . \tag{8}$$

The unit vectors $\boldsymbol{\sigma}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\tau}$ are orthogonal. The scalars a and b are not independent because of a condition that is imposed by solenoidality:

$$\partial\bar{\Omega}_i/\partial r_i=0 ,$$
$$r(\gamma a_r+\gamma_1 b_r)+\gamma_1(\gamma_1 a_\gamma-\gamma b_\gamma)=0 , \tag{9}$$

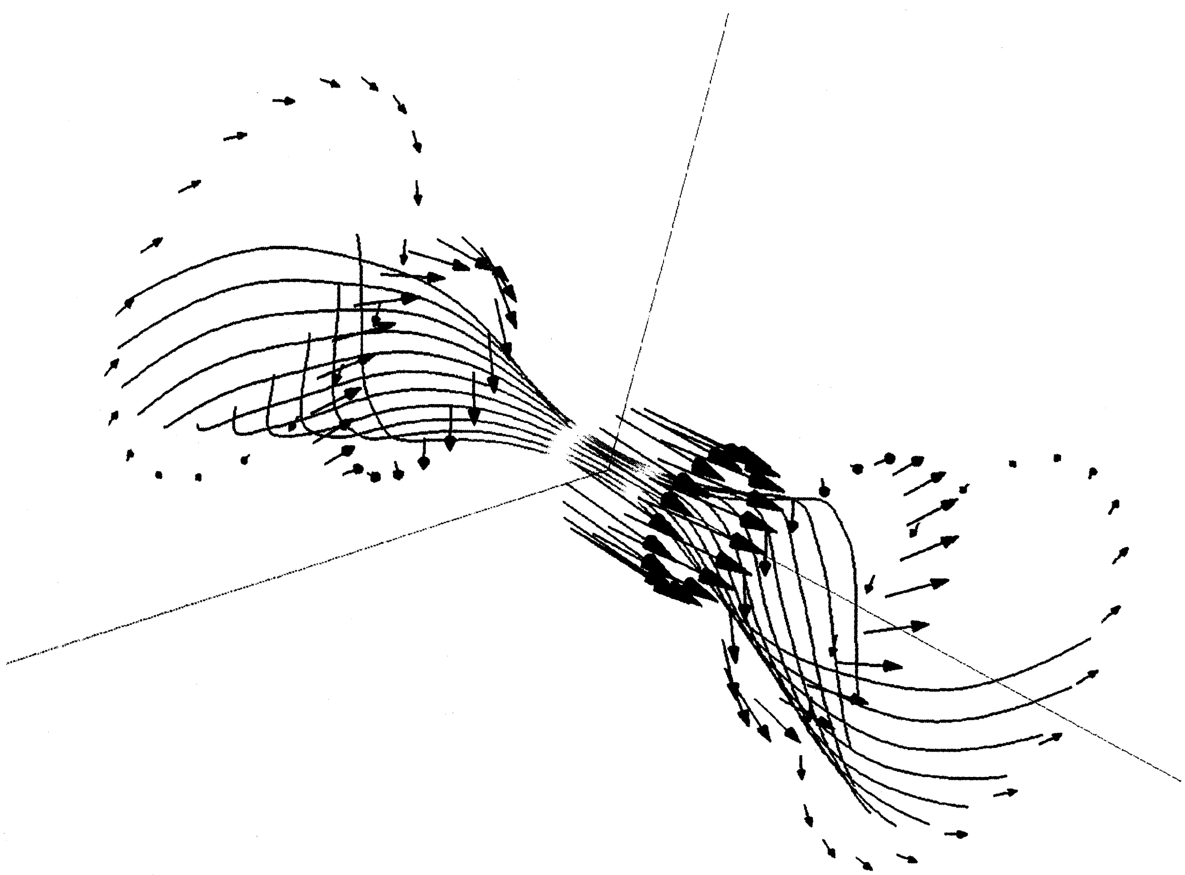

FIG. 1. Conditionally averaged vorticity field. The colors indicate vorticity magnitude. The blue color corresponds to $\bar{\Omega}=0$ and the red color corresponds to $\bar{\Omega}=\omega_*$. The arrows indicate the direction of the vortex lines.

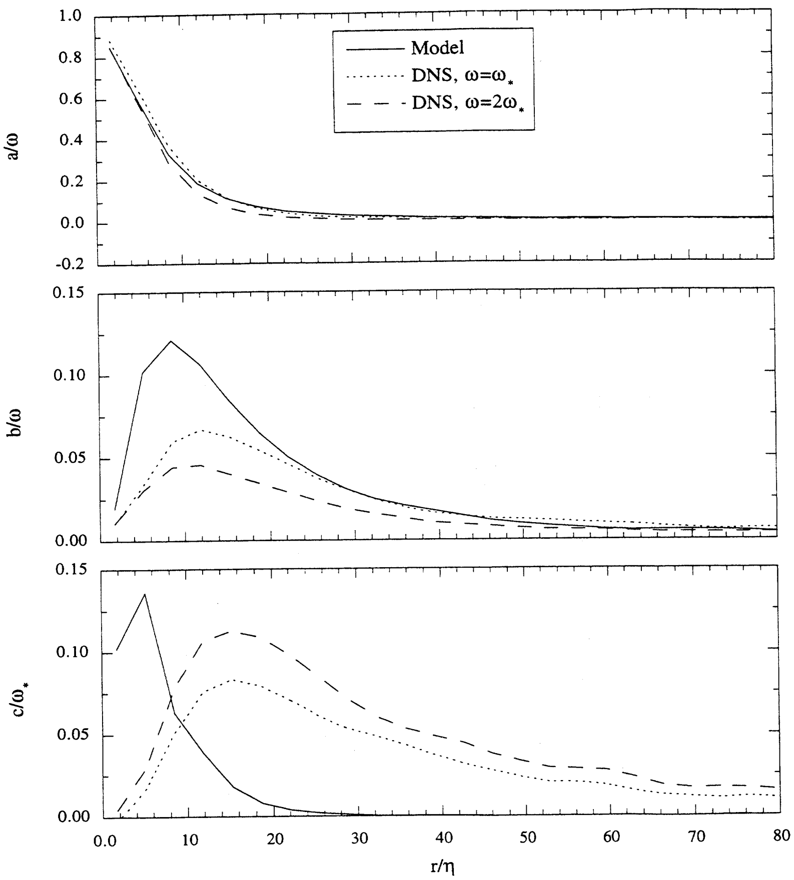

FIG. 2. Scalar coefficients (a,b,c) as functions of radial distance. $\theta=\arccos(\gamma)=45^\circ$ and the radial distance is normalized by the Kolmogorov microscale, $\eta=(\nu^3/\epsilon)^{1/4}$. In accordance with equations (10) and (11), scalars a and b are normalized by ω, and c is normalized by ω_*.

where subscripts indicate differentiation. These two scalars are expressed in terms of one scalar g from (2) and scalar c is expressed in terms of h:

$$\begin{aligned} a &= \int \exp(i\mathbf{k}\cdot\mathbf{r})(1-\mu^2)g\,d^3k \;, \\ b &= -\tfrac{1}{2}\gamma\gamma_1 \int \exp(i\mathbf{k}\cdot\mathbf{r})(3\vartheta^2-1)g\,d^3k \\ &= -\frac{\gamma\gamma_1}{3\gamma^2-1}\int \exp(i\mathbf{k}\cdot\mathbf{r})(3\mu^2-1)g\,d^3k \;, \\ c &= \tfrac{1}{2}\gamma\gamma_1 \int \exp(i\mathbf{k}\cdot\mathbf{r})(3\vartheta^2-1)h\mu^{-1}d^3k \;, \end{aligned} \tag{10}$$

where $\vartheta=n_i\rho_i$.

In order to extract information about these scalars from a DNS data set S of the vorticity field, we first have to determine a subset $S_\omega \in S$, corresponding to grid points with magnitude of vorticity in a bin centered around ω. Summation over all (say, n) such bins gives the whole set: $\sum S_\omega = S$. Next, we choose one point $\mathbf{x} \in S_\omega$. At this point we will have a certain vorticity vector $\boldsymbol{\omega}$ with a known magnitude in the chosen bin. From all other points $\mathbf{x}' \in S$ we choose a subset $S_{r,\gamma}(\mathbf{x},\boldsymbol{\omega})$ of points with distances $r=|\mathbf{x}'-\mathbf{x}|$ in a certain r bin and $\gamma=\rho\sigma$ in a γ bin. This subset depends on the original point $\mathbf{x}$ and the corresponding $\boldsymbol{\omega}$. At every point $\mathbf{x}' \in S_{r,\gamma}(\mathbf{x},\boldsymbol{\omega})$ we will have certain vorticity vector $\boldsymbol{\omega}'$. Now we average $\boldsymbol{\omega}'$ over all points $\mathbf{x}' \in S_{r,\gamma}(\mathbf{x},\boldsymbol{\omega})$. By projecting the resulting vorticity field on vectors $\boldsymbol{\sigma}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\tau}$, we will have a sample of the scalars (a,b,c), which is still dependent on $(\mathbf{x},\boldsymbol{\omega})$. Additional averaging over all $\mathbf{x} \in S_\omega$ gives the CAV field (8). The ergodicity assumption is utilized here by replacing ensemble averaging (1) by space averaging. The DNS of isotropic turbulence is described in Ref. [5].

A CAV field that is obtained by this procedure from DNS data is presented in Fig. 1 for a magnitude of fixed vorticity $\omega=\omega_*$ and a Taylor Reynolds number $R_\lambda \approx 72.4$. The twisting part of vorticity corresponds to two distributed coaxial vortex rings with opposite signs of vorticity, producing stretching of the central fluid element in the direction of $\boldsymbol{\omega}$. The hyperboloidal part of vorticity arises due to viscous smoothing. The characteristic attenuation scale for the twisting and hyperboloidal CAV components is order ten times greater than the Kolmogorov microscale, $\eta=(\nu^3/\varepsilon)^{3/4}$. For turbulent free-surface flows, the twisting part of the CAV field is expected to connect to the free surface. This conjecture is based on experience with laminar and quasilaminar vortex reconnections with free surface [10].

The general topology of this CAV field is basically the same as in a simple analytical model:

$$\begin{aligned} g(k,\omega,\mu) &= (3\nu\omega/2\varepsilon)\phi(k) \;, \\ h(k,\omega,\mu) &= -(15\nu^2k^2\mu/2\varepsilon)\phi(k) \;, \end{aligned} \tag{11}$$

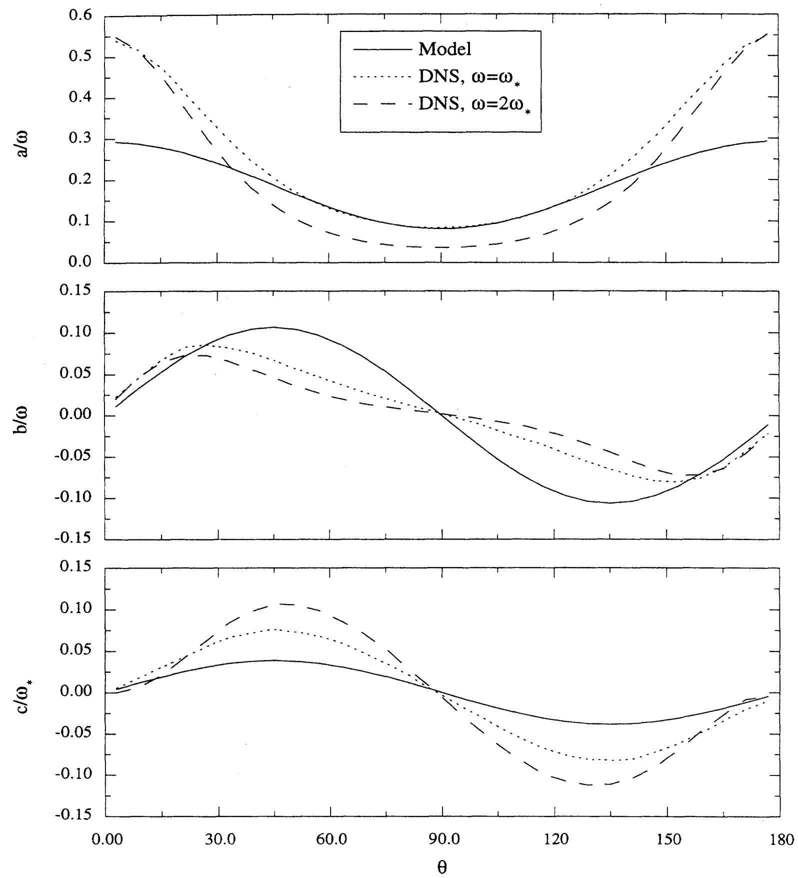

FIG. 3. Scalar coefficients (a,b,c) as functions of angles. $\theta = \arccos(\gamma)$ and $r/\eta = 11.92$.

where $\phi(k) = E(k)/2\pi = \langle \tilde{\Omega}_i \omega_i \rangle$ and $E(k)$ is the energy spectrum. We note that for the Gaussian probability distribution of the vorticity field, the function g is the same as in the model (11), but $h = 0$. The model was obtained [3] under assumption that $\alpha(\omega) = \text{const}$. This assumption is not correct for the DNS data (6). However, the coefficient in the exponent in (6) is not big (~ 0.16) and a qualitative agreement exists with DNS data, but certain details are different. These differences are reflected in the series of plots (Figs. 2 and 3) for the scalars (a,b,c) that are presented for two values of fixed vorticity: $\omega = \omega_*$ and $2\omega_*$. The agreement between the model and DNS is better for scalar a than for scalars b and c. The functions are smooth and we hope in future to get a simple analytical description of CAV that is consistent with the Navier-Stokes equations and with all necessary conditions [3]. In particular, we need an analytical description for the large-eddy simulations (see the next section). The realization of this goal will require CAV analysis over a broad interval of fixed vorticity, which may require DNS data with higher resolution.

IV. CONCLUSION

The CAV field represents the statistical balance between vortex stretching and viscous smoothing in three-dimensional turbulent flow, conditioned by the level of fixed vorticity. This conditional balance is at the heart of small-scale turbulent motion. It was argued [3] that this balance is the only exact information about two-point vorticity statistics, which can be obtained from the hierarchy of equations for the n-point PDF without resorting to closures. Similar behavior is observed in a variety of systems that have strong nonlinear interactions, including strongly turbulent plasma. The balance is topologically represented by the twisting component of CAV (stretching) and the hyperboloidal component (smoothing). For locally isotropic turbulence the decomposition in terms of scalars (8) depends on three arguments: the distance between two points in turbulent flow, the magnitude of vorticity in one point, and the corresponding angle. The simple analytical model (11) only qualitatively describes this dependence and gives the general topological structure of CAV. So the first problem for the future is to find a quantitative analytical description of CAV, which may require additional numerical experiments. This analytical description is important not only for understanding turbulence and other phenomena that have strong nonlinear interactions, but also for a subgrid-scale modeling in many applications.

One way to make large-eddy simulations consistent with CAV fields is to introduce into the Navier-Stokes equations (written in terms of vorticity) a vortex relaxation term:

$$-\tau_s^{-1}[\omega_i(\mathbf{x}) - Q \int m(\mathbf{r}) \overline{\Omega}_i(-\mathbf{r}, \hat{\omega}(\mathbf{x}+\mathbf{r})) d^3 r] \,,$$

where

$$\tau_s \sim L^{2/3}\varepsilon^{-1/3}\mathrm{Re}^{-1/5}$$

and

$$\hat{\omega}_i(\mathbf{x}+\mathbf{r})=\frac{1}{1+\kappa(\mathbf{r})}[\omega_i(\mathbf{x}+\mathbf{r})+\kappa(\mathbf{r})\overline{\Omega}_i(\mathbf{r},\boldsymbol{\omega}(x))] \ . \tag{12}$$

Here τ_s is the relaxation time [4] that is associated with "vortex strings," Q is the solenoidal projection operator, m and κ are weighting coefficients that depend on the numerical scheme, and $\hat{\omega}$ is an intermediate field that is designed for a smooth relaxation. We see that detailed information about the CAV field over a broad range of vorticity and distance is required for numerical simulations. A similar relaxation effect can be introduced into the Navier-Stokes equations written in terms of velocity, but the physics is still based on vorticity. It is assumed that the grid scale is of order of $l_s \sim L\ \mathrm{Re}^{-3/10}$ [4], which gives a huge potential savings in the numerical workload [6]. At this scale the most dangerous (numerically) nonlinear effects of vortex stretching and convection do not produce a flux in the vorticity correlations [4]. Thus, we can expect a relatively smooth connection between numerics and modeling at the scale. We do not know yet if the particular relaxation (12) will work, but in order to test it we need a detailed (preferably analytical) description of the CAV field.

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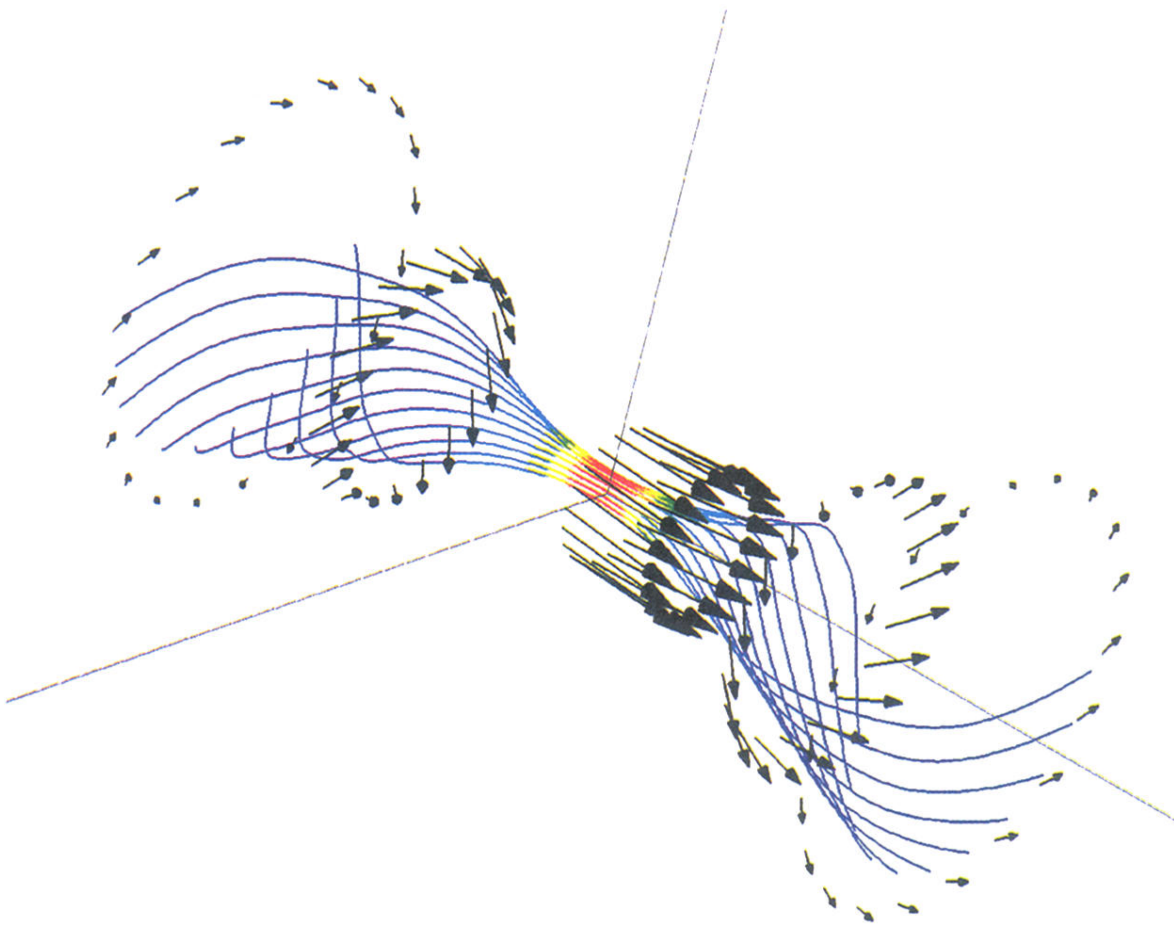

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